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A derivation of the time dependence of the cluster of two spins

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Abstract. Time-dependent perturbation theory is used to calculate the probabilities for the states and their transitions of a cluster of two spins. The time dependence is included in the boundary conditions which are used as described in a recent paper by Bolton and Johnson. The ideas are tested briefly on the simple mean-field case. The structure of the transition probabilities suggested by Suzuki and Kubo is verified with expressions dependent on the nature of the coupling between the lattice and the spins.

1. Introduction

We are concerned here with the approach to equilibrium of an Ising system governed by the master equation for the probability function $P(s_1, \ldots, s_N; t)$. The variables s_i, $i = 1 \ldots N$ are scalars taking the values ± 1 and are the z components of the spin vector operator representing spin-$\frac{1}{2}$. The spin system is in contact with a heat bath which allows spin flips to occur. The transition probabilities are denoted by $W(s_1, \ldots, s_i \to -s_i, \ldots, s_N)$. The master equation for the Ising system was first discussed by Glauber (1963); it is

$$\frac{\mathrm{d}P}{\mathrm{d}t}(s_1, \ldots, s_N; t) = -\sum_i W(s_1, \ldots, s_i \to -s_i, \ldots, s_N) P(s_1, \ldots, s_i, \ldots, s_N; t) + \sum_i W(s_1, \ldots, -s_i \to s_i, \ldots, s_N) P(s_1, \ldots, -s_i, \ldots, s_N; t). \tag{1.1}$$

Glauber considered this equation as the starting point but it must be able to be derived from a Liouville equation.

Heims (1965) obtained the general master equation for the Ising model interacting with a lattice, without bringing in the boundary conditions; he used a different interaction Hamiltonian from the one we shall use and applied his equation to the case of high temperatures, well above transition temperature. Our aim in the present paper is different from Heims; we are concerned with the structures of the probability of the state of the system and of the transition probabilities in the master equation and to this end we have chosen a two-spin system. Essentially we have derived the time dependence of the constant-coupling approximation originally introduced by Kasteleijn and von Kranendonk (1956).

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Suzuki and Kubo (1968) suggested a possible form for the transition probabilities which satisfies the condition of detailed balance in equilibrium and this has been used for example to describe some properties of the evolution toward equilibrium of a system in the Bragg–Williams approximation (Bolton and Leng 1973, 1975) and in the Bethe–Peierls approximation (Batterham *et al* 1976). In these latter calculations the Fokker–Planck equation was derived before the solution was sought but in the present calculations we proceed from the Liouville equation to the master equation.

The reservoir or heat bath is considered to be a set of uncoupled boson or fermion modes with a correlation time much shorter than the time scales of interest in the spin systems, so that we may consider it to remain in thermal equilibrium. The structure of the transition probabilities depends on the structure of the interaction between the reservoir and the spin system. The properties of a two-spin system are defined in terms of the two quantities

$$s(t) = \langle s_1 \rangle = \langle s_2 \rangle \tag{1.2}$$

and

$$\phi(t) = \langle s_1 s_2 \rangle, \tag{1.3}$$

where the averages are calculated in terms of $P(s_1 \ldots s_N, t)$. The two spins s_1, s_2 are nearest neighbours on a plane square lattice with the coordination number $\gamma = 4$ and the usual Ising nearest-neighbour interactions are taken. The spins neighbouring the two selected spins are each assumed to have the self-consistent value of $s(t)$. These boundary conditions are called extended mean-field boundary conditions and were explored in some detail by Bolton and Johnson (1976). Since the direct interaction between the two spins is the full Ising interaction defined by the exchange energy

$$-Js_1 s_2$$

the problem we are considering is the time-dependent constant coupling approximation originally introduced by Kasteleijn and van Kranendonk (1956) to examine the static properties of the Ising model. In § 2 we give a brief survey of the method as applied to a single-site cluster where a single spin s_1 is surrounded by the γ boundary spins each carrying the average spin $\langle s_1 \rangle$. This is just the familiar mean-field approximation and since the solution is already known in the literature (Suzuki and Kubo 1968, Goldstein and Scully 1973, Wang 1973 and Metui *et al* 1975) we present it only briefly as a survey of our method before moving on to the two-site cluster in § 3. The solution by Metui *et al* (1975) in Bragg–Williams approximation has been constructed from that presented here by combining N single-site solutions.

2. The statistical perturbation theory applied to the simple mean-field problem

The Liouville equation is expressed in terms of the density operator $\rho(t)$ and since the reservoir is assumed to be always in thermal equilibrium, we can factorize $\rho(t)$ as follows

$$\rho(t) = \rho_s(t) \otimes f_R, \tag{2.1}$$

where $\rho_s(t)$ refers to spin variables only and f_R is the equilibrium density operator for the reservoir. The spin density matrix remains diagonal in the basis set of unperturbed states of the spin system and this is consistent with the form of the probability function proposed by Glauber (1963) and used by Bolton and Johnson (1976). For one spin,

$$P(s; t) = \tfrac{1}{2}(1 + s_1 s(t)) \tag{2.2}$$

and for two spins,

$$P(s_1, s_2; t) = \frac{1}{2^2}[1 + (s_1 + s_2)s(t) + s_1 s_2 \phi(t)]. \tag{2.3}$$

The notation is that of Goldstein and Scully (1973). The Liouville equation of motion is written in the interaction representation and to second order of perturbation the equation for $\rho_s(t)$ is

$$\frac{\partial \rho_s}{\partial t} = \frac{1}{\hbar^2} \int_{-\infty}^{t} dt' \, \mathrm{Tr}_R(V(t)V(t')\rho_s \otimes f_R$$

$$- V(t)\rho_s \otimes f_R V(t') - V(t')\rho_s \otimes f_R V(t) + \rho_s \otimes f_R V(t')V(t)), \tag{2.4}$$

where

$$V(t) = \exp(iH_0 t/\hbar) V \exp(-iH_0 t/\hbar). \tag{2.5}$$

The probability functions used in the master equation are interpreted as the diagonal elements of $\rho_s(t)$. For example, for the single-site cluster, we have

$$P(s_1 = 1; t) = \langle +1|\rho_s(t)|+1\rangle \tag{2.6}$$

$$P(s_1 = -1; t) = \langle -1|\rho_s(t)|-1\rangle \tag{2.7}$$

and these can be combined into (2.2).

For the single-site cluster we have

$$H = H_0 + V, \tag{2.8}$$

where $H_0 = H_s + H_R$,

$$H_s = -\gamma J s s_1, \tag{2.9}$$

$$H_R = \sum_\alpha \hbar \omega_\alpha b_\alpha^\dagger b_\alpha, \tag{2.10}$$

$$V = \sum_\alpha g_\alpha (s_1^- b_\alpha + s_1^+ b_\alpha^\dagger), \tag{2.11}$$

$$s_1^\pm = s_1^x \pm i s_1^y, \tag{2.12}$$

$$s_1 \equiv s_1^z, \tag{2.13}$$

$$[s_1^x, s_1^y] = 2i s_1^z. \tag{2.14}$$

We include the general symbol for the coordination number: so that we can put $\gamma = 4$ for the plane square and $\gamma = 6$ for the simple cubic. Thus

$$V(t) = \sum_\alpha g_\alpha \{\exp[i(2\gamma J s/\hbar - \omega_\alpha)t] s_1^- b_\alpha + \mathrm{HC}\}. \tag{2.15}$$

Substituting $V(t)$ into (2.4) we obtain for the matrix element $\langle +1|\rho_s(t)|+1\rangle$

$$\frac{\partial}{\partial t}\langle +1|\rho_s(t)|+1\rangle$$

$$= -\frac{1}{\hbar^2} \int_{-\infty}^{t} dt' \left(8 \sum_\alpha g_\alpha^2 \cos(2\gamma J s/\hbar - \omega_\alpha)(t - t')\right)$$

$$\times [n_\alpha \langle +1|\rho_s(t')|+1\rangle - (1 \pm n_\alpha)\langle -1|\rho_s(t')|-1\rangle]$$

with similar equations for the other elements. In the equations for the diagonal matrix elements the right-hand sides contain only diagonal elements since the coefficients of the off-diagonal elements contain factors like $\langle bb\rangle_R$ and $\langle b^\dagger b^\dagger\rangle_R$ which vanish since the reservoir is in equilibrium. For the off-diagonal elements the equation for $\langle +1|\rho_s(t)|-1\rangle$, for example, involves only off-diagonal elements. The equations are homogeneous so that if the off-diagonal matrix elements are zero initially, they remain zero for all time.

To proceed. We assume the reservoir spectrum to be dense (this in effect follows from the assumption of equilibrium) so that the sum over discrete modes may be replaced by an integral over a continuum with density of states $D(\omega)$ and interaction strength $g(\omega)$. We further assume the spectrum to be broad so that the correlation time of the reservoir is very much shorter than the mean time between spin transitions. Integration over ω_α then yields an integrand in t' which is strongly peaked at $t'=t$ with essentially a delta function behaviour. Integration over t' then yields terms which depend on $\rho_s(t')$ for $t'<t$. That is, we have a Markov process.

Thus, we have

$$\frac{\partial P(+1;t)}{\partial t} = -\frac{1}{\hbar^2}8\pi D(\omega)g^2(\omega)\delta(2\gamma Js/\hbar-\omega)[\bar{n}(\omega)P(+1;t)-(1\pm\bar{n}(\omega))P(-1;t)] \tag{2.16}$$

where

$$\bar{n} = \langle b^\dagger b\rangle_R = \mathrm{Tr}_R\{b^\dagger b^\dagger f_R\} = 1/[\exp(\beta\hbar\omega)\mp 1].$$

The upper sign in the expressions refers to a boson reservoir and the lower sign to a fermion reservoir. Equation (2.16) is a master equation of the kind specified in (1.1) and the transition probabilities can readily be seen to be of the form given in Bolton and Johnson with their coefficient $a(s)/2\alpha$ identified as

$$a(s)/2\alpha = 4\pi D(s)g^2(s)B(s)/\hbar^2, \tag{2.17}$$

where $D(s)$ and $g(s)$ have been appropriately re-defined, with

$$B(s) = \begin{cases} \coth as, & \text{bosons} \\ 1, & \text{fermions} \end{cases} \tag{2.18}$$

and

$$a = \beta\gamma J. \tag{2.19}$$

3. The cluster of two sites

We shall treat the interaction between the two spins s_1 and s_2 exactly, and assign the spin $s(t)$ defined by

$$s(t) = \langle s_1\rangle = \langle s_2\rangle \tag{3.1}$$

to the remaining nearest-neighbour sites surrounding this pair. The Hamiltonian is

$$H_0 = H_s + H_R$$

with

$$H_s = -(\gamma-1)Jss_1 - (\gamma-1)Jss_2 - Js_1s_2. \tag{3.2}$$

We assume the particles to interact with the reservoir independently of one another.

This is equivalent to each spin having its own reservoir and we write

$$H_R = \sum_\alpha \hbar\omega_\alpha (b_\alpha^\dagger b_\alpha + a_\alpha^\dagger a_\alpha) \tag{3.3}$$

so that the interaction between spin system and reservoir may be written

$$V = \sum_\alpha g_\alpha (s_1^+ b_\alpha^\dagger + s_2^+ a_\alpha^\dagger) + \text{HC}. \tag{3.4}$$

Further, since the two reservoirs are mutually isolated we may formally write the density matrix for this two open system as

$$\rho(t) = \rho_s(t) \otimes f_{R_1} \otimes f_{R_2};$$

this factorization is to be understood in the sequel. We have also considered the inclusion of the 'rotating wave' terms $s_1^+ b_\alpha + s_2^+ a_\alpha$ and HC but they do not affect the final result.

The perturbation treatment allows us to express the spin system in terms of the basis set of states of the unperturbed Hamiltonian. We use the general notation $|s_1, s_2\rangle$, remembering that s_1, s_2 can only take the values ± 1.

Then,

$$H_0|s_1, s_2, (n_\alpha)\rangle = \left(-(\gamma - 1)J(s_1 + s_2)s - Js_1 s_2 + \sum_\alpha \hbar\omega_\alpha n_\alpha |s_1, s_2, (n_\alpha)\rangle \right) \tag{3.5}$$

and s_1, s_2 and n_α on the right-hand side of (3.5) are eigenvalues. We use the following relations for spin operators and phonon operators:

$$\exp(\alpha s_1) s_1^\pm = \exp(\pm 2\alpha) s_1^\pm \exp(\alpha s_1) \tag{3.6}$$

$$\exp(\alpha b^\dagger b) b = \exp(-\alpha) b \exp(\alpha b^\dagger b) \tag{3.7}$$

$$\exp(\alpha b^\dagger b) b^\dagger = \exp(\alpha) b^\dagger \exp(\alpha b^\dagger b). \tag{3.8}$$

We find in the interaction representation that

$$V(t) = \sum_\alpha \exp\left(-\frac{it}{\hbar}[2(\gamma - 1)Js - \hbar\omega_\alpha] \right) \times \left[b_\alpha^\dagger s_1^+ \exp\left(-\frac{it}{\hbar} 2Js_2 \right) + a_\alpha^\dagger s_2^+ \exp\left(-\frac{it}{\hbar} 2Js_1 \right) \right] + \text{HC}. \tag{3.9}$$

We substitute for $V(t)$ from (3.9) into (2.4) and again consider the matrix elements in the representation $\{|s_1, s_2\rangle\}$. We denote the matrix elements by $\langle s_1 s_2 | \rho_s(t) | s_1 s_2 \rangle$ with the abbreviations $\langle + + | \rho_s(t) | + + \rangle$, etc for the individual elements. Consider for example the matrix elements for the state $|+ +\rangle$. We have

$$\frac{\partial}{\partial t} \langle + + | \rho_s(t) | + + \rangle = -\frac{1}{\hbar^2} \int_{-\infty}^{t} dt' \, \text{Tr}_R(\langle + + | VV' \rho_s f_R - V \rho_s f_R V' - V' \rho_s f_R V + \rho_s f_R V' V | + + \rangle) \tag{3.10}$$

Taking the fourth term in the integrand of (3.10) we have

$$V|++\rangle = \sum_\alpha g_\alpha \exp\{it[2(\gamma-1)Js/\hbar-\omega_\alpha]\}2\exp\left(\frac{it2J}{\hbar}\right)(a_\alpha|+-\rangle+b_\alpha|-+\rangle)$$

and

$$V'V|++\rangle = \sum_\alpha\sum_\beta g_\alpha g_\beta \exp\{it[2(\gamma-1)Js/\hbar-\omega_\alpha]\}\exp\{-it'[2(\gamma-1)Js/\hbar-\omega_\beta]\}$$
$$\times 4(b_\beta^\dagger b_\alpha + a_\beta^\dagger a_\alpha)\exp\left(\frac{i2J}{\hbar}(t-t')\right)|++\rangle$$
$$+(\text{terms involving } b_\alpha a_\alpha,\ a_\alpha b_\alpha)|--\rangle.$$

Taking the trace over the reservoirs the coefficient of $|--\rangle$ vanishes and we have

$$\mathrm{Tr}\langle++|\rho_s f_R V'V|++\rangle$$
$$= \sum_\alpha g_\alpha^2 \exp[\![i\{2[(\gamma-1)Js+1]/\hbar-\omega_\alpha\}(t-t')]\!]\,8\bar{n}_\alpha\langle++|\rho_s(t')|++\rangle.$$

The first term in (3.10) is the Hermitian conjugate of this and together the first and fourth terms combine to give

$$\sum_\alpha g_\alpha^2\, 16\cos[(\omega_+-\omega_\alpha)(t-t')]\bar{n}_\alpha\langle++|\rho_s(t')|++\rangle,$$

where

$$\omega_\pm = 2J[(\gamma-1)s\pm 1]/\hbar. \tag{3.11}$$

Similarly, the second and third terms give

$$-\sum_\alpha g_\alpha^2 8\cos[(\omega_+-\omega_\alpha)(t-t')](1\pm\bar{n}_\alpha)(\langle+-|\rho_s|+-\rangle+\langle-+|\rho_s|-+\rangle),$$

where as usual the upper sign refers to the boson reservoir and the lower sign to the fermion reservoir. As before we assume that the reservoir has a broad dense spectrum so that

$$\sum_\alpha \to \int_{-\infty}^{\infty} d\omega\, D(\omega)$$

and the integral over t' is

$$\int_{-\infty}^{t} dt'\cos[(\omega_+-\omega)(t-t')] = \pi\delta(\omega_+-\omega).$$

Proceeding as in the one-particle case and using the above result, we find that (3.10) reduces to

$$\frac{\partial}{\partial t}\langle++|\rho_s(t)|++\rangle$$
$$= -A(\omega_+)[\bar{n}(\omega_+)2\langle++|\rho_s|++\rangle$$
$$-(1\pm\bar{n}(\omega_+))(\langle+-|\rho_s|+-\rangle+\langle-+|\rho_s|-+\rangle)]. \tag{3.12}$$

Similarly we find

$$\frac{\partial}{\partial t}\langle --|\rho_s|--\rangle$$

$$= -A(\omega_-)[2(1\pm\bar{n}(\omega_-))\langle --|\rho_s|--\rangle$$
$$-\bar{n}(\omega_-)(\langle +-|\rho_s|+-\rangle+\langle -+|\rho_s|-+\rangle)] \tag{3.13}$$

and

$$A(\omega) = 8\pi D(\omega)g^2(\omega)/\hbar^2. \tag{3.14}$$

As in the one-particle case the off-diagonal elements do not appear in (3.13) since the reservoirs remain in equilibrium. We have a similar equation for the diagonal matrix element $\langle -+|\rho_s|-+\rangle$ which, by the symmetry of the cluster, equals $\langle +-|\rho_s|+-\rangle$. For the off-diagonal matrix elements the equations again contain only off-diagonal elements because of the independence of the interactions of spins with their separate reservoirs.

The master equation that we need for this cluster is

$$\frac{\partial}{\partial t}P(s_1, s_2; t) = P(s_1, s_2; t)(-W(s_1, s_2 \to -s_2) - W(s_1 \to -s_1, s_2))$$
$$+P(-s_1, s_2; t)W(-s_1 \to s_1, s_2) + P(s_1, -s_2; t)W(s_1, -s_2 \to s_2). \tag{3.15}$$

We now identify the diagonal matrix elements as state probabilities and write

$$P(s_1, s_2; t) = \langle s_1 s_2|\rho_s(t)|s_1 s_2\rangle. \tag{3.16}$$

A comparison of (3.15) with (3.12) and (3.13) allows us to make the identifications

$$W(1 \to -1, 1) = W(1, 1 \to -1) = A(\omega_+)\bar{n}(\omega_+) = \tfrac{1}{2}A(\omega_+)B(\omega_+)(1-\tanh\tfrac{1}{2}\beta\hbar\omega_+) \tag{3.17}$$

$$W(-1 \to 1, 1) = W(1, -1 \to 1) = A(\omega_+)(1\pm\bar{n}(\omega_+)) = \tfrac{1}{2}A(\omega_+)B(\omega_+)(1+\tanh\tfrac{1}{2}\beta\hbar\omega_+) \tag{3.18}$$

$$W(-1 \to 1, -1) = W(-1, -1 \to 1) = A(\omega_-)(1\pm\bar{n}(\omega_-))$$
$$= \tfrac{1}{2}A(\omega_-)B(\omega_-)(1+\tanh\tfrac{1}{2}\beta\hbar\omega_-) \tag{3.19}$$

$$W(1 \to -1, -1) = W(-1, 1 \to -1) = A(\omega_-)\bar{n}(\omega_-) = \tfrac{1}{2}A(\omega_-)B(\omega_-)(1-\tanh\tfrac{1}{2}\beta\hbar\omega_-), \tag{3.20}$$

where

$$B(\omega) = \begin{cases} \coth\tfrac{1}{2}\beta\hbar\omega, & \text{boson} \\ 1, & \text{fermion.} \end{cases} \tag{3.21}$$

We can assemble equations (3.17)–(3.20) into a single expression

$$W(s_1, s_2 \to -s_2) = \tfrac{1}{2}A(\omega(s_1))B(\omega(s_1))\{1 - s_2 \tanh\beta J[(\gamma-1)s + s_1]\}, \tag{3.22}$$

where we have written $\omega(s_1)$ for the general description of ω_+ and ω_-. We compare this with Suzuki and Kubo's expression for a system of N spins

$$W(s_1, \ldots, s_j \to \ldots, -s_j, \ldots, s) = (1/2\tau)(1 - s_j \tanh\beta E_j), \tag{3.23}$$

where E_j is the local field at the jth site. Suzuki and Kubo comment that their τ is taken to be a constant but may in general depend on the temperature and on spins other than the jth, but do not go further than this. Our calculation verifies this statement and defines the dependence of τ on the properties of the reservoir.

The transition probabilities (3.17)–(3.20) (or (3.22)) can readily be shown to satisfy detailed balance with the equilibrium probability function

$$P_0(s_1, s_2) = \exp(-\beta H_s(s_1, s_2))/\mathrm{Tr}\exp(-\beta H_s(s_1, s_2)).$$

We note that the transition probabilities assumed by Bolton and Johnson (1976) and given in equations (3.18) and (3.19) of that paper can be rewritten in the form

$$W(s_1, s_2 \to -s_2) = (1/2\alpha)\exp[(\gamma-1)\beta J s s_1]\cosh\beta J[(\gamma-1)s + s_1] \times\{1 - s_2\tanh\beta J[(\gamma-1)s + s_1]\}. \tag{3.24}$$

Comparing this expression (3.24) with (3.22) given by the present treatment we see that while (3.24) contains the correct local field factor, the multiplicative factor differs from what we have calculated from the Liouville equation. Both expressions (3.22) and (3.24) agree in describing equilibrium correctly since any multiplicative factors cancel in equilibrium. However the rates of change of the mean order $s(t)$ and two-particle correlation function $\phi(t)$ will depend on these multiplicative factors and the time-dependent treatment given for these in Bolton and Johnson has to be revised in the way presented in § 4.

4. Equations of motion for $s(t)$ and $\phi(t)$

The equations of motion are obtained by substituting the expression (2.3) for $\rho_s(t)$ in equations (3.12) and (3.13). These can be solved for $\mathrm{d}s/\mathrm{d}t$ and $\mathrm{d}\phi/\mathrm{d}t$ to give, for a boson reservoir,

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{-A(\omega)}{\cosh 2(\gamma-1)\beta Js - \cosh 2\beta J} \times[s\sinh 2(\gamma-1)\beta Js - \phi\sinh 2\beta J - \cosh 2(\gamma-1)\beta Js + \cosh 2\beta J] \tag{4.1}$$

and

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{2A(\omega)}{\cosh 2(\gamma-1)\beta Js - \cosh 2\beta J} \times\{s[\cosh 2(\gamma-1)\beta Js - \exp(-2\beta J)] - \phi\sinh 2(\gamma-1)\beta Js\}. \tag{4.2}$$

For a fermion reservoir,

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{-A(\omega)}{\cosh 2(\gamma-1)\beta Js + \cosh 2\beta J} \times\{s[\cosh 2(\gamma-1)\beta Js + \exp(-2\beta J)] - \sinh 2(\gamma-1)\beta Js\} \tag{4.3}$$

and

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{2A(\omega)}{\cosh 2(\gamma-1)\beta Js + \cosh 2\beta J} \times\{s\sinh 2(\gamma-1)\beta Js - \phi[\cosh 2(\gamma-1)\beta Js + \cosh 2\beta J] + \sinh 2\beta J\}. \tag{4.4}$$

In deriving (4.1)–(4.4) we have assumed that both the coupling constant $g(\omega)$ and the density of states $D(\omega)$ of the reservoir are sufficiently slowly varying so that we can write

$$A(\omega_+) \approx A(\omega_-) = A(\omega).$$

To get the equilibrium solutions, we set $\mathrm{d}/\mathrm{d}t = 0$ in (4.1)–(4.4), and define $s(\infty) = \sigma$, $\phi(\infty) = \eta$. An alternative way of getting the result is through the condition for equilibrium detailed balance, namely

$$P_0(1, 1)W(1, 1 \to -1) = P_0(1, -1)W(1, -1 \to 1) \tag{4.5}$$

$$P_0(-1, -1)W(-1, -1 \to 1) = P_0(-1, 1)W(-1, 1 \to -1) \tag{4.6}$$

and by using (2.3) in equilibrium together with (3.17)–(3.20). Both ways agree in giving

$$\sigma = \sinh 2(\gamma - 1)\beta J\sigma / [\cosh 2(\gamma - 1)\beta J\sigma + \exp(-2\beta J)] \tag{4.7}$$

$$\eta = [\cosh 2(\gamma - 1)\beta J\sigma - \exp(-2\beta J)] / [\cosh 2(\gamma - 1)\beta J\sigma + \exp(-2\beta J)], \tag{4.8}$$

which when $\gamma = 4$ are the same as the expressions (3.22) and (3.23) of Bolton and Johnson (1976) calculated from the partition function for the two-site cluster.

It will be noticed that the multiplicative functions on the right-hand side of (4.1) and (4.2) for the boson reservoir can be singular. This is not a fundamental weakness because it arises from the structure of the cluster which mirrors the infinite lattice and it implies that the influence of one of the spins being treated exactly can overweigh the influence on the other spin of the mean field from the boundary spins. A similar point was noticed by Bolton and Gruen (1976) in their numerical simulation of small Ising clusters. As the cluster size increases the effect of this difficulty will decrease. Further work is in progress on the analysis of the approach to equilibrium.

5. Conclusion

We have shown how time-dependent perturbation yields the dynamical solution for the spin cluster of two sites. Essentially it is the time-dependent constant coupling method. In the limit $t \to \infty$ the long range order $s(t)$ and nearest-neighbour correlation function $\phi(t)$ are exactly the values σ, η respectively calculated from the equilibrium partition function.

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